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An Extremal Property for Chebyshev Polynomials

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In this article, it is proved that the Chebyshev polynomials are also the least deviation functions in the class

$$\mathscr{F}_n := \{f \in C^{(n)}_{[-1,1]} : f^{(n)}(x) \geqslant 1,\ x \in [-1, 1]\}$$

with some norms. © 1992 Academic Press, Inc.

1. INTRODUCTION AND NOTATION

It is well known that in the class of polynomials of degree n with leading coefficient 1 the Chebyshev polynomials have the least deviation from 0 on $[-1, 1]$ with some norms (see, e.g. [2]). This article will generalize this result. We first need some definitions and notation which will be used throughout the article.

When $1 < q < \infty$, $\|f\|_q$ denotes the L_q-norm of function f with Chebyshev weight function $1/\sqrt{1-x^2}$; i.e.,

$$\|f\|_q = \left(\int_{-1}^{1} |f(x)|^q \frac{dx}{\sqrt{1-x^2}}\right)^{1/q}.$$

When $q = 1$ or ∞, $\|f\|_q$ stands for the usual L_q-norm of f, so

$$\|f\|_1 = \int_{-1}^{1} |f(x)|\, dx \qquad \text{and} \qquad \|f\|_\infty = \max_{|x| \leqslant 1} |f(x)|.$$

Define $\mathscr{F}_n$ to be the class of all $C^{(n)}_{[-1,1]}$ functions with nth derivatives greater than or equal to 1; i.e.,

$$\mathscr{F}_n = \{f \in C^{(n)}_{[-1,1]} : f^{(n)}(x) \geqslant 1,\ x \in [-1, 1]\}.$$

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With respect to the norm $\|\cdot\|_q$ $(1 \leqslant q \leqslant \infty)$, define the least deviation in the class $\mathscr{F}_n$ as

$$E_q(n) = \inf_{f \in \mathscr{F}_n} \|f\|_q.$$

In this article we consider the solution of extremal problem. It is found that for $1 < q \leqslant \infty$, in the class $\mathscr{F}_n$ the unique extremal function is $T_n(x)/2^{n-1}n!$, where $T_n(x)$ is the Chebyschev polynomial of the first kind; i.e.,

$$T_n(x) = \cos(n \arccos x).$$

For the norm $\|\cdot\|_1$, the unique extremal function is $U_n(x)/2^{n-1}n!$, where $U_n(x)$ is the Chebyshev polynomial of the second kind defined as

$$U_n(x) = \sin[(n+1) \arccos x]/\sin(\arccos x).$$

Therefore the Chebyshev polynomials also have the least deviation from 0 on $[-1, 1]$ in the class $\mathscr{F}_n$. The main results are reported in Section 2 and all proofs are given in Section 3.

2. Main Results

With the notation of Section 1, we state the least deviation theorems and their corollaries in this section with respect to the norms $\|\cdot\|_\infty$, $\|\cdot\|_1$, and $\|\cdot\|_q$ $(1 < q < \infty)$, respectively.

THEOREM 1. *$f^*(x) = T_n(x)/2^{n-1}n!$ is the unique function in $\mathscr{F}_n$ such that*

$$\|f^*\|_\infty = E_\infty(n) = 1/2^{n-1}n!.$$

The following corollary is immediate.

COROLLARY 1. *If $f^{(n)}(x)$ is continuous and does not vanish on $[-1, 1]$, then*

$$\|f\|_\infty \geqslant \frac{1}{2^{n-1}n!} \min_{|x| \leqslant 1} |f^{(n)}(x)|$$

with equality if and only if $f = cT_n$, c being any nonzero constant.

THEOREM 2. *$f^*(x) = U_n(x)/2^{n-1}n!$ is the unique function in $\mathscr{F}_n$ such that*

$$\|f^*\|_1 = E_1(n) = 1/2^{n-1}n!.$$

COROLLARY 2. *If $f^{(n)}(x)$ is continuous and does not vanish on $[-1, 1]$, then*

$$\|f\|_1 \geqslant \frac{1}{2^{n-1}n!} \min_{|x| \leqslant 1} |f^{(n)}(x)|$$

with equality if, and only if, $f = cU_n$, c being any nonzero constant.

THEOREM 3. *$f^*(x) = T_n(x)/2^{n-1}n! \in \mathscr{F}_n$ is the unique function in $\mathscr{F}_n$ such that*

$$\|f^*\|_q = E_q(n) = \frac{1}{2^{n-1}n!}\left[B\left(\frac{1}{2}, \frac{q+1}{2}\right)\right]^{1/q},$$

where $B(\cdot, \cdot)$ is the beta function.

COROLLARY 3. *If $f^{(n)}(x)$ is continuous and does not vanish on $[-1, 1]$, then*

$$\|f\|_q \geqslant \frac{[B(1/2, (q+1)/2)]^{1/q}}{2^{n-1}n!} \min_{|x| \leqslant 1} |f^{(n)}(x)|$$

with equality if and only if $f = cT_n$, c being any nonzero constant.

3. PROOFS OF THEOREMS

To prove Theorem 1, we need the following lemmas.

LEMMA 1. *If $f \in \mathscr{F}_n$ with $\|f\|_\infty = 1/2^{n-1}n!$, then $f(x_k) = (-1)^k/2^{n-1}n!$, where $x_k = \cos(k\pi/n)$, for $k = 0, 1, ..., n$.*

Proof. Write $\omega(x) = (x - x_0) \cdots (x - x_n)$. Consider a sum

$$S = \sum_{k=0}^{n} \frac{\|f\|_\infty - (-1)^k f(x_k)}{|\omega'(x_k)|}.$$

Each term in the sum is non-negative, so $S \geqslant 0$. On the other hand,

$$S = \|f\|_\infty \sum_{k=0}^{n} \frac{1}{|\omega'(x_k)|} - \sum_{k=0}^{n} \frac{f(x_k)}{\omega'(x_k)} = \frac{1}{n!} - f[x_0, x_1, ..., x_n], \tag{1}$$

where $f[x_0, x_1, ..., x_n]$ is the divided difference of f at points $x_0, x_1, ..., x_n$. We know that there exists a $\xi \in [-1, 1]$ such that

$$f[x_0, x_1, ..., x_n] = f^{(n)}(\xi)/n!. \tag{2}$$

Noting that $f^{(n)}(\xi) \geqslant 1$, (1) together with (2) gives $S \leqslant 0$. Therefore, S must be 0 so that all terms in the sum S must be 0, implying that $f(x_k) = (-1)^k \|f\|_\infty = (-1)^k/2^{n-1}n!$, for $k = 0, 1, ..., n$. ▮

LEMMA 2. *Let $g(x)$ be such that $g^{(n)}(x) \geqslant 0$. If $g(x_k) = 0$ at $n+1$ distinct points $1 \geqslant x_0 > x_1 > \cdots > x_n \geqslant -1$, then $g(x) = 0$ for all $x \in [x_n, x_0]$.*

Proof. The lemma is proved by induction. It is obvious when $n = 1$. Suppose that $g^{(k+1)}(x) \geqslant 0$ and $g(x)$ has $k+2$ distinct zeros $1 \geqslant x_0 > x_1 > \cdots > x_{k+1} \geqslant -1$. Write $f(x) = g'(x)$. So $f^{(k)}(x) \geqslant 0$ and $f(x)$ has $k+1$ distinct zeros that interlace with the zeros of g. Let x'_k be the smallest zero of $f(x)$ such that $x'_k \geqslant x_{k+1}$ and x'_0 is the largest zero of $f(x)$ such that $x'_0 \leqslant x_0$. (x'_k and x'_0 are well-defined.) By the induction assumption, $f(x) = 0$ for $x \in [x'_k, x'_0]$. Thus $g(x) = 0$ for $x \in [x'_k, x'_0]$. Since $f(x)$ does not vanish in two intervals (x_{k+1}, x'_k) and (x'_0, x_0) by the definitions of x'_k and x'_0, $g(x)$ is monotone in the two intervals, so $g(x)$ must be equal to 0 in the two intervals since $g(x_{k+1}) = g(x_0) = 0$. Consequently, $g(x) = 0$ on $[x_{k+1}, x_0]$, finishing the induction. ▮

Proof of Theorem 1. That $f^*(x) = T_n(x)/2^{n-1}n!$ is an extremal function so that $E_\infty(n) = 1/2^{n-1}n!$ is a direct result of the Bernstein theorem (see, e.g., [1, p. 38]) can be seen by noting that $f^{*(n)}(x) \equiv 1$ and

$$E_\infty(n) = \inf_{f \in \mathscr{F}_n} \|f\|_\infty = \inf_{f \in \mathscr{F}_n} \|f - p_{n-1}\|_\infty$$

for any polynomial p_{n-1} of degree $n-1$.

Suppose now that there exists another function $f \in \mathscr{F}_n$ such that $\|f\|_\infty = E_\infty(n) = 1/2^{n-1}n!$. Let $g(x) = f(x) - f^*(x)$. Then $g^{(n)}(x) \geqslant 0$. Applying Lemma 1, $g(x_k) = 0$, where $x_k = \cos(k\pi/n)$ for $k = 0, ..., n$. By Lemma 2, $g(x) \equiv 0$ on $[x_n, x_0] = [-1, 1]$, i.e., $f(x) \equiv f^*(x)$ on $[-1, 1]$. Uniqueness is proved. ▮

To prove Theorem 2, we need the following lemmas.

LEMMA 3. $\|f^*\|_1 = E_1(n) = 1/2^{n-1}n!$, *where* $f^*(x) = U_n(x)/2^{n-1}n!$.

Proof. Let $x_k = \cos[k\pi/(n+1)]$, for $k = 1, ..., n$. Consider the interpolating polynomial of f at the n points $p_{n-1}(x) = \sum_{k=1}^n f(x_k)\, l_{n,k}(x)$, where $l_{n,k}(x)$ is the Lagrange polynomial of degree $n-1$, i.e.,

$$l_{n,k}(x) = \frac{(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}.$$

There is a $\xi \in (x_n, x_1)$ such that

$$f(x) - p_{n-1}(x) = \frac{(x-x_1)\cdots(x-x_n)}{n!} f^{(n)}(\xi). \tag{3}$$

Noting that $E_1(n) = \inf_{f \in \mathscr{F}_n} \|f\|_1 = \inf_{f \in \mathscr{F}_n} \|f - p_{n-1}\|_1$ and using (3) we have

$$E_1(n) \geqslant \int_{-1}^{1} \left| \frac{(x-x_1)\cdots(x-x_n)}{n!} \right| dx = \int_{-1}^{1} |f^*(x)| \, dx = 1/2^{n-1}n!.$$

But $f^* \in \mathscr{F}_n$, so the lemma follows. ∎

LEMMA 4. *Let* $f \in \mathscr{F}_n$ *with* $\|f\|_1 = E_1(n)$ *and write* $f^*(x) = U_n/2^{n-1}n!$. *Then* (i) $f(x)$ *and* $f^*(x)$ *have the same signs and zeros, and* (ii) *the derivative of* $\hat{f} = f - f^*$ *has at least n distinct zeros in* $(-1, 1)$.

Proof. By Lemma 3, $\|f\|_1 = \|f^*\|_1$. Since $(f + f^*)/2 \in \mathscr{F}_n$, we have that $\|(f + f^*)/2\|_1 \geqslant (\|f\|_1 + \|f^*\|_1)/2$, i.e.,

$$\int_{-1}^{1} \{|f(x) + f^*(x)| - |f(x)| - |f^*(x)|\} \, dx \geqslant 0. \tag{4}$$

But the integrand in (4) is non-positive and continuous, so it vanishes on $[-1, 1]$, i.e., $|f(x) + f^*(x)| = |f(x)| + |f^*(x)|$ for any $x \in [-1, 1]$. This shows that $f(x)$ and $f^*(x)$ must have the same signs and zeros, proving (i).

To prove (ii), note that $x_k = \cos[k\pi/(n+1)]$, for $k = 1, ..., n$, are n distinct zeros of $f^*(x)$, so are the zeros of $\hat{f}(x)$. By Rolle's theorem, there are at least $n-1$ zeros of $\hat{f}'(x)$ that interlace with x_k's. Next we can show that if $\hat{f}(x)$ has no additional zeros then some x_k's are the zeros of $\hat{f}'(x)$ and hence (ii) follows.

To the contrary, suppose that $\hat{f}(x)$ has no additional zeros and $\hat{f}'(x_k) \neq 0$ for all $k = 1, ..., n$. Then all x_k's are single zeros of $\hat{f}(x)$, and $\hat{f}(x)$ changes signs whenever it passes those zeros. Recall that $f(x)$ and $f^*(x)$ have the same signs. Hence we get that either $\|f\|_1 < \|f^*\|_1$ or $\|f\|_1 > \|f^*\|_1$. This is a contradiction to $\|f\|_1 = E_1(n) = \|f^*\|_1$. ∎

Proof of Theorem 2. Lemma 3 indicates that $f^*(x) = U_n(x)/2^{n-1}n!$ is an extremal function. Suppose now that there exists another $f \in \mathscr{F}_n$ such that $\|f\|_1 = E_1(n)$. Consider $\hat{f}(x) = f(x) - f^*(x)$ and $g(x) = \hat{f}'(x)$. Since $f^{*(n)}(x) \equiv 1$, $g^{(n-1)}(x) \geqslant 0$. By part (ii) of Lemma 4, $g(x)$ has at least n distinct zeros. Let x_{L} and x_{U} be the smallest and largest zeros of $g(x)$ on $[-1, 1]$. In the light of Lemma 2, $g(x) = 0$ for all $x \in [x_{\mathrm{L}}, x_{\mathrm{U}}]$ and hence $\hat{f}(x)$ is constant on $[x_{\mathrm{L}}, x_{\mathrm{U}}]$. By part of (i) of Lemma 4, within $[x_{\mathrm{L}}, x_{\mathrm{U}}]$

there are at least n zeros of $\hat{f}(x)$. It follows that $f(x)=f^*(x)$ for every $x\in[x_L, x_U]$.

Next we study the end subintervals $I_0=[-1, x_L)$ and $I_1=(x_U, 1]$. From the definitions of x_L and x_U, $\hat{f}(x)$ are monotone in I_0 and I_1, respectively. For $x\in I_0$ or I_1 there is a ξ such that

$$\hat{f}(x)=\frac{\hat{f}^{(n)}(\xi)}{n!}(x-x_L)^n.$$

Since $\hat{f}^{(n)}(\xi)\geqslant 0$, $\hat{f}(x)$ and $(x-x_L)^n$ have the same signs. Note that $f^*(-1)=(-1)^n(n+1)$ and $f^*(1)=n+1$, and that by part (i) of Lemma 4, $f(x)$ and $f^*(x)$ have the same signs. Therefore, if n is even and $x\in I_0$ or I_1, $\hat{f}(x)>0$ and hence $f(x)>f^*(x)>0$ so that $\|f\|_1>\|f^*\|_1$, which is a contradiction. If n is odd, then for $x\in I_0$, $\hat{f}(x)<0$ and hence $f(x)<f^*(x)<0$, and for $x\in I_1$, $\hat{f}(x)>0$ and hence $f(x)>f^*(x)>0$. Consequently, for odd n we also get the contradiction that $\|f\|_1>\|f^*\|_1$. The proof is completed. ∎

Proof of Theorem 3. Similarly to the proof of Lemma 3, we can see that

$$\inf_{f\in\mathscr{F}_n}\|f\|_q=\frac{1}{2^{n-1}n!}\|T_n\|_q. \tag{5}$$

On the other hand,

$$\|T_n\|_q^q=B\left(\frac{1}{2},\frac{q+1}{2}\right).$$

This together with (5) proves that $f^*(x)$ is the extremal function and gives the value of $E_q(n)$.

To prove the uniqueness, suppose that $f\in\mathscr{F}_n$ is such that $\|f\|_q=E_q(n)$. Thus $\|f\|_q=\|f^*\|_q$. Since $(f+f^*)/2\in\mathscr{F}_n$,

$$\|(f+f^*)/2\|_q\geqslant\|f\|_q=(\|f\|_q+\|f^*\|_q)/2.$$

But Minkowski inequality gives the inverse inequality so that we have

$$\|f+f^*\|_q=\|f\|_q+\|f^*\|_q. \tag{6}$$

The strictly convex property of norm $\|\cdot\|_q$ when $1<q<\infty$ together with (6) implies that $f=cf^*$ with certain constant c. It is clear that $f^{(n)}(x)=c\geqslant 1$. And since $\|f\|_q=c\|f^*\|_q=\|f^*\|_q$, $c=1$, as desired. ∎

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